RAMIFICATION IN THE COHOMOLOGY OF ALGEBRAIC SURFACES ARISING FROM ORDINARY DOUBLE POINT SINGULARITIES

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Abstract. Let K be a field complete with respect to a discrete valuation whose residue field is perfect of an odd positive characteristic. We study the ramification in the cohomology of a smooth proper surface X defined over K, under the assumption that X admits an integral model $\mathcal{X}$ whose special fibre has at worst ordinary double points. We will introduce a numerical invariant of $\mathcal{X}$, in terms of which the ramification in the cohomology of X is determined.

1. Introduction

Let K be a field which is complete with respect to a discrete valuation. Assume that its residue field k is perfect of characteristic $p > 0$. We fix an algebraic closure $\overline{K}$ of K, and let G_K be the Galois group $\mathrm{Gal}(\overline{K}/K)$. There is a short exact sequence

$$1 \to I_K \to G_K \to G_k \to 1$$

where I_K, called the inertia subgroup of G_K, is the largest subgroup of G_K which acts trivially on the residue field of $\overline{K}$. The quotient G_k is naturally identified with the absolute Galois group of k.

When X/K is an algebraic variety and ℓ is a prime, the etale cohomology group $H^i(X/\overline{K}, \mathbb{Q}_\ell)$ is naturally equipped with a G_K-action. The analysis of this G_K-action is of a great interest, which is illustrated by the celebrated theorem of Serre-Tate [12] and Neron-Ogg-Shafarevich [9].

Theorem (Serre-Tate, Neron-Ogg-Shafarevich)**.** *Let A/K be an abelian variety and ℓ be any fixed prime different from p. The action of I_K on $H^1(A/\overline{K}, \mathbb{Q}_\ell)$ is trivial if and only if A has good reduction.*

When X is a smooth proper variety over K, we say X has good reduction if there is a smooth proper map $\mathcal{X} \to \mathrm{Spec}\,(R)$ between algebraic spaces, where R is the valuation ring of K, whose generic fibre is isomorphic to X. The above theorem is established for elliptic curves by Neron-Ogg-Shafarevich, and for abelian varieties by Serre-Tate.

More generally, if X is a smooth proper variety which has good reduction then all of its ℓ-adic etale cohomology groups with $\ell \neq p$ are unramified. However, the converse is not true, to which curves of genus at least two are counterexamples. The implication of the form

$$H^i(X/\overline{K}, \mathbb{Q}_\ell) \text{ is unramified for all } i \implies X \text{ has good reduction}$$

being false, one naturally considers its variations.

First, one may strengthen the assumption so that the good reduction of X can be deduced. This approach is taken by T. Oda [8], where it was established that if X is a curve then the triviality of the outer action of I_K on suitable quotients of the etale fundamental group of X implies that X has good reduction. Its p-adic analogue has been established by Andreatta-Iovita-Kim [1]. In a similar spirit, Chiarellotto-Lazda-Liedtke [2] considered K3 surfaces which potentially admit a so-called Kulikov model, and showed that the unramifiedness of $H^i(X/\overline{K}, \mathbb{Q}_\ell)$ together with vanishing of a certain nonabelian cohomology class implies good reduction.

Second, one may consider a particular type of varieties, and try to show the converse. For example, Liedtke-Matsumoto [6] considered a K3 surface which admits a potential semistable model, and established that the unramifiedness of $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ implies the good reduction of X over an unramified extension of K.

Third, one may weaken the conclusion, by aiming at something weaker than good reduction. A prominent result in this direction is due to Rapoport-Zink [10]. They start with an integral model $\mathcal{X}/R$ which is semistable, which gives rise to a filtration, called the weight filtration, of $H^i(X/\overline{K}, \mathbb{Q}_\ell)$. The weight-monodromy conjecture says that I_K acts trivially on the graded quotients of the weight filtration. Conditionally on the weight-monodromy conjecture, the cohomology $H^i(X/\overline{K}, \mathbb{Q}_\ell)$ is unramified if and only if the weight filtration on it has at most one non-trivial graded quotient. If the characteristics of K and k are equal, then the weight-monodromy conjecture is known in full generality by Ito [5] who reduced it to the cases established by Deligne [3]. Although the weight-monodromy conjecture is not fully known in the mixed characteristic case, it is established for surfaces by Rapoport-Zink [10] and for set theoretic complete intersections in a smooth projective toric variety by Scholze [11]. Their p-adic analogues also exist, notably due to Hyodo-Kato [4] and Mokrane [7].

In this short note, we consider a combination of the second and third variations, in that the relation between the I_K-action on $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ and properties of integral models of a smooth surface X is investigated. Instead of starting from a semistable model, we will assume, with a modest goal in mind, the existence of an integral model $\mathcal{X}$ of X such that the special fibre $\mathcal{X}_k$ of $\mathcal{X}$ has at worst ordinary double points. An ordinary double point is the singularity at the origin of the affine equation $z_0z_1 + z_2^2 = 0$, which is arguably the mildest singularity that can occur in a surface. We further assume that $p \nmid 2\ell$.

To each singular point x of $\mathcal{X}_k$, we will define a numerical invariant n_x, a positive integer, which is determined by the formal neighborhood of x in $\mathcal{X}$. Using this, we define an invariant

$$g(\mathcal{X}) \in \mathbb{Z}_{\geq 0},$$

a nonnegative integer, to be the number of singular points x such that n_x is odd.

On the Galois theoretic side, we will consider quadratic characters; surjective homomorphisms $G_K \twoheadrightarrow \{\pm 1\}$. A quadratic character will be called ramified if its restriction to I_K is non-trivial. Under the assumption $p \nmid 2$, there is a unique ramified quadratic character, which we will denote throughout by ψ. Recall that a vector space with semisimple G_K-action is ψ-isotypic if any non-trivial irreducible G_K-submodule is isomorphic to ψ.

Our main theorem relates $g(\mathcal{X})$ to the I_K-action on $H^2(X/\overline{K}, \mathbb{Q}_\ell)$.

Theorem. *Assume $p \nmid 2\ell$. The codimension of $H^2(X/\overline{K}, \mathbb{Q}_\ell)^{I_K}$ in $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ is equal to $g(\mathcal{X})$. Furthermore, the quotient $H^2(X/\overline{K}, \mathbb{Q}_\ell)/H^2(X/\overline{K}, \mathbb{Q}_\ell)^{I_K}$ is ψ-isotypic for the I_K-action.*

We sketch its proof. The crucial ingredients are the Rapoport-Zink spectral sequence and the weight-monodromy theorem. In order to apply these tools, we first construct a (potential) semistable model $\mathcal{X}^{\mathrm{ss}}$ of X. More precisely, we construct a particular semistable model $\mathcal{X}^{\mathrm{ss}}/R_L$ of X/L, where L/K is the unique ramified quadratic extension of K with valuation ring R_L. This semistable model further enjoys the property that the Galois action of $\mathrm{Gal}(L/K)$ on X/L extends to $\mathcal{X}/R_L$ in a way that it is compatible with its action on R_L. The analysis of the $\mathrm{Gal}(L/K)$-action on the Rapoport-Zink spectral spectral sequence for $\mathcal{X}^{\mathrm{ss}}/R_L$ will yield the above theorem.

The techniques employed in the above sketch of proof are rather similar to those in [6]. The authors of [6] investigated K3 surfaces X/K such that it has a potential semistable reduction over some Galois extension K'/K. Although such a smooth model $\mathcal{X}$ of X/K' may not admit an action of $\mathrm{Gal}(K'/K)$, they showed that there is always a birational modification $\mathcal{X}'$ of $\mathcal{X}$, such that $\mathrm{Gal}(K'/K)$ acts on $\mathcal{X}'$, and that $\mathcal{X}'_k$ has at worst rational double points.

Note that the ordinary double point is the simplest kind among the rational double points, so in terms of the singularities involved our scope is more limited. We have a relative advantage to be able to construct an explicit semistable model over L with Galois action.

What is not attempted in the present article includes two natural questions. One is whether our method can be generalized to rational double points. The other is to find an approach which does not rely on the assumption $p > 2$, whence it works for all residue characteristics.

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2. Statement of main result

2.1. **A numerical invariant $g(\mathcal{X})$.** We would like to define a numerical invariant of an integral model $\mathcal{X}$ of a smooth proper surface X/K. In fact, it is determined by $X/\breve{K}$ where $\breve{K}$ is the completion of the maximal unramified extension of K. We will assume, in this subsection, that k is separably closed, and that $K = \breve{K}$.

For every singular point x of $\mathcal{X}_k$, the formal neighborhood of x in $\mathcal{X}$ has the coordinate ring isomorphic to

$$A := \frac{R[[z_0, z_1, z_2]]}{z_2^2 + z_0 z_1 + r} \tag{2.1.1}$$

with some $r \in \mathfrak{p}$. Because X is smooth, r is unequal to zero. Let n_x be the $\mathfrak{p}$-adic order of r, so $r \in \mathfrak{p}^{n_x}$ and $r \notin \mathfrak{p}^{n_x+1}$. In particular, for any ordinary double point $x \in \mathcal{X}_k$, we have $n_x \in \mathbb{Z}$ and $n_x \geq 1$.

Proposition 2.1.2. *The number n_x is well-defined.*

Proof. Consider the sheaf $\Omega^1_{\mathcal{X}/R}$ of relative differentials of $\mathcal{X} \to \operatorname{Spec} R$. Let I_x be the stalk of the zeroth Fitting ideal of $\Omega^1_{\mathcal{X}/R}$. We claim that, if $\mathcal{O}_x$ is the local ring of germs of functions near x, then n_x is equal to the length of $\mathcal{O}_x/I_x$ as an $\mathcal{O}_x$-module.

From (2.1.1), one can present $\Omega^1_{A/R}$ as the A-module generated by dz_i, $i = 0, 1, 2$, subject to the relation $z_0 dz_1 + z_1 dz_0 + 2z_2 dz_2 = 0$. Its zeroth Fitting ideal is therefore generated by z_0, z_1, and $2z_2$. Since we are assuming that $p \nmid 2$, the last generator $2z_2$ may be replaced with z_2 without affecting the ideal. The quotient $B = A/(z_0, z_1, z_2)$ has length n_x as an A-module, because the filtration $0 = \pi^{n_x} B \subset \pi^{n_x - 1} B \subset \cdots \subset B$ yields n_x irreducible subquotients of B.

To conclude, this provides us with an intrinsic definition of n_x. □

Definition 2.1.3. Let X be a smooth surface with an integral model $\mathcal{X}$ such that $\mathcal{X}_k$ has at worst ordinary double points. We define $g(\mathcal{X})$ to be the number of singular points x in X_k such that n_x is odd.

Without assuming $K = \breve{K}$, we simply define $g(\mathcal{X})$ to be $g(\mathcal{X}/\breve{R})$, where $\breve{R}$ is the valuation ring of $\breve{K}$.

2.2. **The statement.** Here we state our main theorem and its corollaries, whose proofs will be given in § 3. We keep the assumptions of the previous subsection. In particular, $K = \breve{K}$ and $\mathcal{X}$ is an integral model of a smooth surface X/K, such that $\mathcal{X}_k$ has at worst ordinary double points. Our main theorem is the following.

Theorem 2.2.1. *Assume $p \nmid 2\ell$. The codimension of $H^2(X/\overline{K}, \mathbb{Q}_\ell)^{I_K}$ in $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ is equal to $g(\mathcal{X})$. Furthermore, the quotient $H^2(X/\overline{K}, \mathbb{Q}_\ell)/H^2(X/\overline{K}, \mathbb{Q}_\ell)^{I_K}$ is ψ-isotypic for the I_K-action.*

As an immediate corollary, we obtain:-

Corollary 2.2.2. *Suppose that $\mathcal{X}$ and $\mathcal{X}'$ are two integral models of a smooth surface X/K such that their special fibres have at worst ordinary double points. Then we have $g(\mathcal{X}) = g(\mathcal{X}')$.*

3. The proof of the main theorem

We will prove Theorem 2.2.1 in this section. We will first construct a (potential) semistable model, from which the theorem will follow by invoking the Rapoport-Zink spectral sequence and the weight-monodromy theorem.

3.1. Constructing a good semistable model.

Let π be a fixed uniformizer of R. Consider the algebra

$$A_n = R[z_0, z_1, z_2]/(z_2^2 + z_0 z_1 + \pi^n) \tag{3.1.1}$$

for some positive integer n, and let $\mathcal{Z}_n/R$ be the formal neighborhood of the singular point $(z_0, z_1, z_2) = (0, 0, 0)$ in the reduction $A_n/\pi A_n$ of (3.1.1). Our aim is to describe a potential semistable model of $\mathcal{Z}_n$. In fact, our semistable model will be defined over $L = K(\sqrt{\pi})$, which is the field cut out by the unique ramified quadratic character ψ of G_K. Our semistable model will have an additional feature that the Galois group $\mathrm{Gal}(L/K)$ acts on it in a way that the action extends the natural Galois action on the generic fibre, and that the action is compatible with its action on R_L, the valuation ring of L.

We denote respectively the generic and the special fibre of $\mathcal{Z}_n$ by Z_n and $\overline{Z}_n$. Let $x \in \overline{Z}_n(k)$ be the singular point with coordinate $(0, 0, 0)$. Let $\mathrm{Bl}_x\mathcal{Z}_n$ be the blowup of $\mathcal{Z}_n$ at x.

Proposition 3.1.2. *Suppose that $n > 2$. Then the special fibre of $\mathrm{Bl}_x\mathcal{Z}_n$ is reduced and has two irreducible components which intersect transversally along a projective line. One component is the minimal resolution of $\overline{Z}_n$. The other component is a singular quadric and has an ordinary double point whose formal neighborhood in $\mathrm{Bl}_x\mathcal{Z}_n$ is isomorphic to $\mathcal{Z}_{n-2}$.*

Proof. Let $I_n \subset A_n$ be the ideal generated by z_0, z_1, z_2, and π. The Rees algebra A'_n of I_n has equation

$$A'_n = A_n[w_0, w_1, w_2, w_3]/I'_n \tag{3.1.3}$$

where I'_n is generated by the determinants of 2×2 minors of the matrix

$$\begin{bmatrix} z_0 & z_1 & z_2 & \pi \\ w_0 & w_1 & w_2 & w_3 \end{bmatrix} \tag{3.1.4}$$

together with

$$z_0 w_1 + z_2 w_2 + \pi^{n-1} w_3, \;\; w_0 w_1 + w_2^2 + \pi^{n-2} w_3^2. \tag{3.1.5}$$

We describe the irreducible components of the special fibre of $\mathrm{Bl}_x\mathcal{Z}_n$. There are two components, that are respectively defined by $z_0 = z_1 = z_2 = 0$ and $w_3 = 0$.

The component defined by $z_0 = z_1 = z_2 = 0$ is equal to the blowup of $\overline{Z}_n$ at its singular point, whence is equal to the minimal resolution of it.

The other component is a singular quadric, whose homogeneous ring of coordinates is $k[w_0, w_1, w_2, w_3]/(w_0w_1 + w_2^2)$.

The chart with $w_3 \neq 0$ has the coordinate ring

$$\frac{R[w_0, w_1, w_2]}{(w_0w_1 + w_2^2 + \pi^{n-2})} \tag{3.1.6}$$

which has a unique singular point in the special fibre whose neighborhood is $\mathcal{Z}_{n-2}$. The chart with $w_2 \neq 0$ has the coordinate ring

$$\frac{R[w_0, w_1, w_3, z]}{(w_3z_2 - \pi, w_0w_1 + w_3^2\pi^{n-2} + 1)} \tag{3.1.7}$$

which is semistable. The chart with $w_1 \neq 0$ has the coordinate ring

$$\frac{R[w_0, w_3, z_2]}{(w_3z_2 + \pi)}, \tag{3.1.8}$$

and the chart with $w_0 \neq 0$ has the coordinate ring

$$\frac{R[w_1, w_3, z_0]}{(w_3z_0 + \pi)}, \tag{3.1.9}$$

both of which are semistable. □

Proposition 3.1.10. *Suppose that $n = 2$. Then the special fibre of $\mathrm{Bl}_x\mathcal{Z}_n$ is reduced and has two irreducible components which intersect transversally. One component is the minimal resolution of $\overline{Z}_n$. The other component is isomorphic to a smooth quadric surface.*

Proof. We first note that the computation of Rees algebra in the proof of Proposition 3.1.2 is valid for $n = 2$. In particular, $\mathrm{Bl}_x\mathcal{Z}_n$ is represented by (3.1.3) with $n = 2$, and its affine charts are given by (3.1.6), (3.1.7), (3.1.8), and (3.1.9). In particular, the component of the special fibre defined by $z_0 = z_1 = z_2 = 0$ has the (homogeneous) coordinate ring

$$\frac{k[w_0, w_1, w_2, w_3]}{(w_0w_1 + w_2^2 + w_3^2)} \tag{3.1.11}$$

which defines a smooth quadric surface. □

Proposition 3.1.2 shows that a semistable model for $\mathcal{Z}_n$ can be obtained from that of $\mathcal{Z}_{n-2}$. Proposition 3.1.10 shows that for even n, a semistable model of $\mathcal{Z}_n$ can be found by iterated blowups.

When n is odd, we pass to the quadratic extension L of K. We consider the special case $n = 1$.

Proposition 3.1.12. *Suppose that $n = 1$. The special fibre of $\mathrm{Bl}_x(\mathcal{Z}_1/R_L)$ is reduced and has two irreducible components which intersect transversally. The special fibre of $\mathrm{Bl}_x(\mathcal{Z}_1/R_L)$ is decomposed into $\overline{Z}_{1,0} \cup \overline{Z}_{1,1}$ where $\overline{Z}_{1,0}$ is the minimal resolution of $\overline{Z}_1$ and $\overline{Z}_{1,1}$ is a smooth quadric surface defined by (3.1.11). The Galois group $\mathrm{Gal}(L/K)$ acts on $\mathrm{Bl}_x\mathcal{Z}_1/R_L$ in the manner that it acts trivially on $\overline{Z}_{1,0}$ and acts on $\overline{Z}_{1,1}$ by sending w_3 to $-w_3$.*

Proof. Without loss of generality, we choose π_L to be a uniformizer of L such that $\pi_L^2 = \pi$. Then, $\mathcal{Z}_1/R_L$ has the form

$$z_2^2 + z_0 z_1 + \pi_L^2 = 0 \tag{3.1.13}$$

to which Proposition 3.1.10 applies. Moreover, the center of blowup is defined by the ideal (z_0, z_1, z_2, π_L), which is stable under the action of $\mathrm{Gal}(L/K)$. Hence the Galois group $\mathrm{Gal}(L/K)$ acts on $\mathrm{Bl}_x(\mathcal{Z}_1/R_L)$ in a way that it extends the Galois action on X/L, and that it is compatible with its action on R_L. By Proposition 3.1.10, $Z_{1,1}$ is defined by

$$w_2^2 + w_0 w_1 + w_3^2 = 0 \tag{3.1.14}$$

where the variable w_i for $i = 0, 1, 2$ corresponds to z_i and w_3 corresponds to π_L. The unique nontrivial element of $\mathrm{Gal}(L/K)$ fixes w_i for $i = 0, 1, 2$, and sends w_3 to $-w_3$. □

Now we are ready to construct semistable models $\mathcal{Z}_n^{\mathrm{ss}}$ of $\mathcal{Z}_n$. We consider even and odd n's separately. Let m be any positive integer and let $n = 2m$. We obtain $\mathcal{Z}_n^{\mathrm{ss}}$ by iteratively blowing up the ordinary double point in the singular fibre m-times. When $n = 2m - 1$, then we pass to $\mathcal{Z}_n/R_L$ and iteratively blow up the ordinary double point in the special fibre n-times to get $\mathcal{Z}_n^{\mathrm{ss}}/R_L$. The following propositions describe the special fibres of $\mathcal{Z}_n^{\mathrm{ss}}/R_L$.

Proposition 3.1.15. *Let n be any positive and even integer, say $n = 2m$. The semistable R-model $\mathcal{Z}_n^{\mathrm{ss}}$ of $\mathcal{Z}_n$ has $m + 1$ components in its special fibre. One of them is the minimal resolution of $\overline{Z}_n$, which we denote by $\overline{Z}_{n,0}$. One can write*

$$\overline{Z}_n^{\mathrm{ss}} = \overline{Z}_{n,0} \cup \overline{Z}_{n,1} \cup \cdots \cup \overline{Z}_{n,m} \tag{3.1.16}$$

where $Z_{n,i}$ is the component introduced at the i-th blowup for each $i \geq 1$. For each $i \geq 1$, $\overline{Z}_{n,i}$ is isomorphic to a smooth quadric. The dual graph of $\overline{Z}_n^{\mathrm{ss}}$ is a line segment, and the intersection of two adjacent components is isomorphic to a projective line.

Proof. We apply Proposition 3.1.2 $(m - 1)$-times and apply Proposition 3.1.10. For $i = m$, $\overline{Z}_{n,m}$ is a smooth quadric by Proposition 3.1.10. For $1 \leq i < m$, invoking Proposition 3.1.2, we observe that $\overline{Z}_{n,i}$ is blowup of a singular quadric. The resulting blowup is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, or, equivalently, to a smooth quadric. □

Proposition 3.1.17. *Let n by any positive and odd integer, say $n = 2m - 1$. The semistable R_L-model $\mathcal{Z}_n^{\mathrm{ss}}$ has an $\mathrm{Gal}(L/K)$-action which extends the Galois action on the generic fibre in a way which is compatible with its action on R_L. Its special fibre has has $n + 1$ components. One of them is the minimal resolution of $\overline{Z}_n$, which we denote by $\overline{Z}_{n,0}$. One can write*

$$\overline{Z}_n^{\mathrm{ss}} = \overline{Z}_{n,0} \cup \overline{Z}_{n,1} \cup \cdots \cup \overline{Z}_{n,n} \tag{3.1.18}$$

where $Z_{n,i}$ is the component introduced at the i-th blowup for each $i \geq 1$. In particular, $\overline{Z}_{n,i}$ is isomorphic to a smooth quadric surface for each $i \geq 1$. The dual graph of $\overline{Z}_n^{\mathrm{ss}}$ is a line segment, and the intersection of two adjacent components is isomorphic to a projective line. For each $i \geq 1$, $\overline{Z}_{n,i}$ is isomorphic to a smooth quadric. For all i, $\overline{Z}_{n,i}$ is stable under the action of $\mathrm{Gal}(L/K)$. *If $i < n$, the action is trivial. If $i = n$, the action is nontrivial, which is equivalent to the action of* $\mathrm{Gal}(L/K)$ *on $\overline{Z}_{1,1}$ in Proposition 3.1.12.*

Proof. It is similar to the proof of Proposition 3.1.15. □

3.2. **Rapoport-Zink spectral sequence.** We review the Rapoport-Zink spectral sequence $E_1^{\bullet,\bullet}$ associated to $\mathcal{X}^{\mathrm{ss}}$, with an emphasis on the terms that contribute to $H^2(X/\overline{K}, \mathbb{Q}_\ell)$. Let n be the number of irreducible components of $\overline{X}^{\mathrm{ss}}$, and let V_i for $i = 1, 2, \cdots, n$ be its irreducible components. Note that if i_1, i_2, i_3 are distinct, then $V_{i_1} \cap V_{i_2} \cap V_{i_3} = \emptyset$. Also note that for all distinct pairs (i, j), the cohomology of $V_i \cap V_j$ is concentrated in even degrees, because $V_i \cap V_j$ is either empty or isomorphic to the projective line. These two facts about $\mathcal{X}^{\mathrm{ss}}$ greatly simplify the spectral sequence. Indeed, the part of the spectral sequence which computes $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ is given by a 3-term sequence

$$\bigoplus_{1 \leq i < j \leq n} H^0(V_i \cap V_j) \xrightarrow{d} \bigoplus_{1 \leq i \leq n} H^2(V_i) \xrightarrow{d'} \bigoplus_{1 \leq i < j \leq n} H^2(V_i \cap V_j) \tag{3.2.1}$$

where cohomology of an empty space is regarded as zero. The three terms are respectively $E_1^{-1,2}$, $E_1^{0,2}$, and $E_1^{1,2}$. The first differential d is given by the map $H^0(V_i \cap V_j) \to H^2(V_\mu)$ which is the Gysin map if $i = \mu$ or $j = \mu$, and zero otherwise. Note that in this case the image of the Gysin map is generated by the cycle class of $V_i \cap V_j$. The second differential d' is the sum of maps $H^2(V_\mu) \to H^2(V_i \cap V_j)$, which is the pullback if $i = \mu$, the negation of the pullback if $j = \mu$, and zero otherwise.

Proposition 3.2.2. *We have a natural* $\mathrm{Gal}(\overline{K}/K)$*-equivariant isomorphism*

$$\mathrm{Ker}(d')/\mathrm{Im}(d) \cong H^2(X/\overline{K}, \mathbb{Q}_\ell). \tag{3.2.3}$$

Furthermore, the action of $\mathrm{Gal}(\overline{K}/L)$ *on* $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ *is trivial.*

Proof. The $\mathrm{Gal}(\overline{K}/L)$-equivariant isomorphism is due to Rapoport-Zink. The triviality of the $\mathrm{Gal}(\overline{K}/L)$-action follows from the weight-monodromy conjecture, which is known for surfaces over a mixed characteristic local field by Rapoport-Zink [10] and for all varieties over an equal characteristic local field by Ito [5].

Since the semistable model has an action of $\mathrm{Gal}(L/K)$, it induces an action on the sequence (3.2.1). The functoriality of the Rapoport-Zink spectral sequence implies that this action induces the action of $\mathrm{Gal}(L/K)$ on $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ via the above isomorphism. □

Now we would like to analyze the action of $\mathrm{Gal}(L/K)$ on $H^2(X/\overline{K}, \mathbb{Q}_\ell)$. In order to do that, we compute the Galois action on $E_1^{i,2}$ for $i = -1, 0, 1$. Note that $\mathrm{Gal}(\overline{K}/L)$ acts trivially on $E_1^{i,2}$ for all $i = -1, 0, 1$, and it remains to determine the action of $\mathrm{Gal}(L/K)$.

Proposition 3.2.4. *The action of* $\mathrm{Gal}(L/K)$ *is trivial on* $E_1^{-1,2}$ *and* $E_1^{1,2}$.

Proof. It follows from the observation that the action of $\mathrm{Gal}(L/K)$ is trivial on $V_i \cap V_j$ for every i, j with nonempty intersection. Indeed, if we let V_i be defined by the equation $w_2^2 + w_0 w_1 + w_3^2 = 0$, on which $\mathrm{Gal}(L/K)$ acts by sending w_3 to $-w_3$. Our construction of the semistable model shows that the subvariety $V_i \cap V_j \subset V_i$ is defined by $w_2^2 + w_0 w_1 + w_3^2$ and $w_3 = 0$, and it follows that the action of $\mathrm{Gal}(L/K)$ on $V_i \cap V_j$ is trivial. □

It remains to determine the action of $\mathrm{Gal}(L/K)$ on $E_1^{0,2}$. Recall that ψ is the unique quadratic character of G_K which induces the isomorphism $\mathrm{Gal}(L/K) \xrightarrow{\sim} \{\pm 1\}$.

Proposition 3.2.5. *The dimension of the* ψ*-isotypic component of* $E_1^{0,2}$ *is equal to* $g(\mathcal{X})$.

Proof. Let x be an ordinary double point of $\mathcal{X}_k$ with odd n_x. Let m be the positive integer with $n_x = 2m - 1$. By our construction of $\mathcal{X}^{\mathrm{ss}}$, there is a natural map $q\colon \mathcal{X}_k^{\mathrm{ss}} \to \mathcal{X}_k$, $q^{-1}(x)$ is the union of m smooth quadric surfaces. Among the m smooth quadric surfaces, there is a unique surface, say V_x, on which $\mathrm{Gal}(L/K)$ nontrivially, and the action is described in Proposition 3.1.12. As a $\mathrm{Gal}(L/K)$-representation, $H^2(V_x)$ is isomorphic to $\mathbf{1} \oplus \psi$, where $\mathbf{1}$ is the trivial character. Therefore the dimension of the ψ-isotypic component of $E_1^{0,2}$ is at least $g(\mathcal{X})$.

On the other hand, if an irreducible component V_i of $\mathcal{X}_x^{\mathrm{ss}}$ is not equal to V_x for some x with odd n_x, then the action of $\mathrm{Gal}(L/K)$ on V_i is trivial. Hence the dimension of the ψ-isotypic component of $E_1^{0,2}$ is exactly $g(\mathcal{X})$. □

We can finally connect $g(\mathcal{X})$ to $H^2(X/\overline{K}, \mathbb{Q}_\ell)$.

Proposition 3.2.6. *The dimension of the* ψ*-isotypic component of* $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ *is equal to* $g(\mathcal{X})$.

Proof. It follows from combining Proposition 3.2.5 with Proposition 3.2.2 and Proposition 3.2.4. □

The main theorem in the introduction follows from the above proposition.

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