LOW-DIMENSIONAL ABELIAN VARIETIES
PROBLEM SESSION (JUNE 5, 2019)

Problem 1 (Kiran Kedlaya). Let $X_3 \subseteq \mathbb{P}^{n+1}_{\mathbb{F}_q}$ be a smooth cubic hypersurface. For which $n \in \mathbb{Z}_{\geq 2}$ and q prime power is it guaranteed that X contains a line (over $\mathbb{F}_q$)?

This is known for $n = 2$ and all q by the theory of cubic surfaces: the answer is negative for all q. For $n \geq 5$, the answer is always affirmative by Debarre–Laface–Roulleau. For $n = 3, 4$, there are open cases.

If the answer is not always affirmative, we could ask more generally for the *probability* for fixed n, q, or say as $q \to \infty$. (Wanlin Li)

The probability ought to be be 1 or nearly so for q sufficiently large. We already know this for $n = 5$ and above, so only $n = 3$ and $n = 4$ are at issue. The Fano variety of lines must have rational points for q large enough, except possibly for very special cubics for which the variety decomposes into Galois conjugates—can that happen at all? (Noam Elkies)

Problem 2 (Isabel Vogt). Let $X_3 \subseteq \mathbb{P}^{n+1}_{\mathbb{F}_q}$ be a smooth cubic hypersurface and let $S \subseteq X(\mathbb{F}_q)$ be a finite subset of rational points (with $\#S \leq q + 1$). Does there exist nonconstant $f \colon \mathbb{P}^1_{\mathbb{F}_q} \to X$ such that $S \subseteq f(\mathbb{P}^1(\mathbb{F}_q))$?

It is known if q is large enough for fixed n.

For $\#S = 1$ and cubic surfaces, it is unknown (for $q \leq 8$).

Problem 3 (Vladimir Dokchitser). Let E be an elliptic curve over $\mathbb{Q}$. Let $K \supseteq \mathbb{Q}$ be a cyclic cubic extension and $L \supseteq K$ a cyclic extension with $[L : K] = 21$, and suppose $L \supseteq \mathbb{Q}$ is Galois and nonabelian. Let $\chi\colon \operatorname{Gal}(L \,|\, K) \hookrightarrow \mathbb{C}^\times$ and consider the function $L(E_K, \chi, s)$ obtained as the L-function of E base-changed to K twisted by χ.

Conjectures imply that $3 \mid \operatorname{ord}_{s=1} L(E_K, \chi, s)$. Using modularity of E, can you prove this?

Problem 4 (Chris Rasmussen). There are several places where existing Sage code for solving S-unit equations could be optimized—please help!

A target application would be to list all genus 2 curves over $\mathbb{Q}$ with good reduction away from 3 (Drew Sutherland); it might be more tractable to do those with at least one rational Weierstrass point (Noam Elkies).

Problem 5 (Yuri Zarhin, 1989). Let X be a smooth projective variety over a number field K. Is there a positive density set of primes $\mathfrak{p}$ over K such that $X \bmod \mathfrak{p}$ is ordinary, i.e., the Newton polygon of the action of $\operatorname{Frob}_{\mathfrak{p}}$ acting on $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ is equal to the Hodge polygon for all i.

This is known for projective spaces, elliptic curves, abelian surfaces, Mumford abelian fourfolds, CM abelian varieties (in all dimensions), K3 surfaces, and as well as products of these.

Is there a single example of an absolutely simple, typical abelian threefold for which this is known? (Kiran Kedlaya)

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Problem 6 (Ben Smith). Consider the directed graph G_1 with vertices given by supersingular elliptic curves $\overline{\mathbb{F}}_p$ up to isomorphism and directed edges given by 2-isogenies. The graph G_1 is connected.

Now consider the graph G_2 with vertices principally polarized superspecial abelian surfaces over $\overline{\mathbb{F}}_p$ (i.e., isomorphic to a product of supersingular elliptic curves in an unpolarized way) with edges $(2,2)$-isogenies. Is this graph connected? This has been checked for $p \leq 1000$.

Are the graphs Ramanujan? More generally, what are the spectral properties of G_2? (Noam Elkies)

For G_1, the graph is known to be connected by strong approximation for quaternions. Does this work for G_2?

For E ordinary over $\overline{\mathbb{F}}_p$, consider the graph whose vertices are given by principal polarizations on $E \times E$ and directed edges given by $(2,2)$-self isogenies. This graph is connected.

Problem 7 (Davide Lombardo). Let A be a abelian surface over a number field K that is typical ($\mathrm{End}(A_{\overline{K}}) = \mathbb{Z}$). By Serre, for each prime ℓ, the ℓ-adic Galois representation $\rho_\ell \colon G_K \to \mathrm{GSp}_4(\mathbb{Z}_\ell)$ has open image. Can you bound the index effectively in terms the height of A, $[K : \mathbb{Q}]$, d_K?

When ℓ is large, one can see that the the index is 1 (and there is a formula for this).

Does it help if $K = \mathbb{Q}$? (Nils Bruin)

Problem 8 (Bjorn Poonen). Is there a reasonably practical algorithm to compute the geometric endomorphism ring of an abelian variety over finitely generated fields K, such as global function fields? Such an algorithm exists for finite fields and number fields.

In general, because of the relation between Néron–Severi groups and endomorphism rings of abelian varieties, a paper by Poonen, Testa, and van Luijk implies the existence of a horribly slow algorithm for computing endomorphism rings.

Problem 9 (Edgar Costa). Let A be an abelian variety over a number field. Is there a reasonable algorithm that can access the zeta function of A modulo $\mathfrak{p}$ for arbitrary $\mathfrak{p}$ and the height of A (or the conductor of the L-function) and returns as output the geometric endomorphism algebra?

Work of Chris Hall may be helpful (Jeff Achter).